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Note

A geometric description of the spin-embedding of symplectic dual polar spaces of rank 3

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Abstract

We give a geometrical description of the spin-embedding e_{sp} of the symplectic dual polar space $\Delta \cong DW(5, 2^r)$ by showing how the natural embedding of $W(5, 2^r)$ into $\mathrm{PG}(5, 2^r)$ is involved in the Grassmann-embedding e_{gr} of Δ. We prove that the map sending every quad of Δ to its nucleus realizes the natural embedding of $W(5, 2^r)$. Taking the quotient of e_{gr} over the space spanned by the nuclei of the quadrics corresponding to the quads of Δ gives an embedding isomorphic to e_{sp}.

Keywords: Symplectic dual polar space; Grassmann-embedding; Spin-embedding

1. Introduction

1.1. Basic definitions

Let Π be a thick polar space of rank $n \geqslant 2$ and let Δ be its associated dual polar space. We denote the point-set of Δ by P. For all points x and y of P, $\mathrm{d}_\Delta(x, y)$ denotes the distance between x and y in the collinearity graph of Δ. For every point $x \in P$ and every $i \in \mathbb{Z}$, we define $\Delta_i(x) := \{y \in P \mid \mathrm{d}_\Delta(x, y) = i\}$, $\Delta_i^*(x) := \{y \in P \mid \mathrm{d}_\Delta(x, y) \leqslant i\}$. A subspace S of Δ is called *convex* if every point on a shortest path between two points of S is contained in S.

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The convex subspaces of diameter 2 are called *quads*. The dual polar space Δ is a near polygon (Shult and Yanushka [22], De Bruyn [13]) which means that for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x. More generally, for every point x and every convex subspace S, there exists a unique point $\pi_S(x)$ of S called the *projection of x onto S* such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_S(x)) + \mathrm{d}(\pi_S(x),y)$ for every point y of S. A *hyperplane* of Δ is a proper subspace meeting each line. By Shult [21, Lemma 6.1], every hyperplane of Δ is a maximal subspace. Since Δ is a near polygon, the set $H_x := \Delta^*_{n-1}(x)$ of points at non-maximal distance from a given point x is a hyperplane of Δ. The hyperplane H_x is called the *singular hyperplane with deepest point x*.

A *full projective embedding* of Δ into a projective space Σ is an injective mapping e from the point-set P of Δ to the point-set of Σ satisfying:

(E1) the image $e(P)$ of e spans Σ;
(E2) every line of Δ is mapped by e onto a line of Σ.

The numbers $\dim(\Sigma)$ and $\dim(\Sigma)+1$ are respectively called the *projective* and *vector dimensions* of e. If e is a full embedding of Δ, then for every hyperplane α of Σ, $H_\alpha := e^{-1}(e(P)\cap\alpha)$ is a hyperplane of Δ. We say that the hyperplane H_α *arises from the embedding e*. If every singular hyperplane of Δ arises from e, then e is called *polarized*. If e is polarized, then for every point x of Δ, $\langle e(H_x)\rangle$ is a hyperplane of Σ (recall that H_x is a maximal subspace of Δ).

Two full embeddings $e_1:\Delta\to\Sigma_1$ and $e_2:\Delta\to\Sigma_2$ of Δ are called *isomorphic* ($e_1\cong e_2$) if there exists an isomorphism $f:\Sigma_1\to\Sigma_2$ such that $e_2 = f\circ e_1$. Let $e:\Delta\to\Sigma$ be a full embedding of Δ and U a subspace of Σ satisfying:

(C1) $\langle U, e(p)\rangle \neq U$ for every point p of Δ;
(C2) $\langle U, e(p_1)\rangle \neq \langle U, e(p_2)\rangle$ for any two distinct points p_1 and p_2 of Δ.

Then there exists a full embedding e/U of Δ into the quotient space Σ/U, mapping each point p of Δ to $\langle U, e(p)\rangle$. If $e_1:\Delta\to\Sigma_1$ and $e_2:\Delta\to\Sigma_2$ are two full embeddings, then we say that $e_1\geqslant e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U\cong e_2$. If Δ is embeddable, then by Tits [23, 8.6] and Kasikova and Shult [17, 4.6], Δ admits the so-called *absolutely universal embedding $\tilde{e}$*. The inequality $\tilde{e}\geqslant e'$ holds for every full embedding e' of Δ. By Cardinali, De Bruyn and Pasini [6, Corollary 1.8], this absolutely universal embedding $\tilde{e}$ is polarized.

For every full polarized embedding $e:\Delta\to\Sigma$, let R_e be the intersection of all hyperplanes $\langle e(H_x)\rangle$, $x\in P$, of Σ. Then R_e satisfies conditions (C1) and (C2), and the embedding $\bar{e} := e/R_e$ is a full polarized embedding of Δ which is called the *minimal full polarized embedding* of Δ. If e_1 is another full polarized embedding of Δ, then $e_1\geqslant\bar{e}$ and $\bar{e}_1\cong\bar{e}$. For proofs and more information on the above facts, we refer to Cardinali, De Bruyn and Pasini [5].

An embedding $e:\Delta\to\Sigma$ is called *homogeneous* if every automorphism of Δ lifts to an automorphism of Σ. Among all full embeddings of Δ, the homogeneous ones are the most interesting. Examples of homogeneous embeddings are the minimal full polarized embeddings [5], the absolutely universal embeddings (Kasikova and Shult [17]; Ronan [20]), the Grassmann-embeddings (Cooperstein [8,9]; De Bruyn [14,15]) and the spin-embeddings (Chevalley [7]; Cooperstein and Shult [10]).

1.2. Notation and main results

Let $n \geqslant 2$ and let $V = V(2n, q)$ denote a $2n$-dimensional vector-space over a finite field $\mathbb{F}_q$ equipped with a symplectic form $(\cdot,\cdot)$. The subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ define a symplectic polar space Π. The corresponding dual polar space Δ is denoted by $DW(2n-1, q)$ and is called a symplectic dual polar space. Denote by $\perp$ the symplectic polarity associated to $(\cdot,\cdot)$. If x is a point of Π, $x^\perp$ denotes the set of points of Π collinear with or equal to x in Π.

Throughout this paper, we assume $q = 2^r$ and $\Delta := DW(2n-1, q)$. Then $DW(2n-1, q) \cong DQ(2n, q)$, where $DQ(2n, q)$ is the dual polar space associated to $Q(2n, q)$.

Let $\bigwedge^n V$ be the nth exterior power of V which is a $\binom{2n}{n}$-dimensional vector space over $\mathbb{F}_q$. If α is a maximal totally isotropic subspace of V and if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ is a collection of vectors generating α, then let $\wedge(\alpha)$ denote the 1-dimensional subspace $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $\bigwedge^n V$. So $\wedge^n$ defines a map from the point-set of Δ to the point-set of $\mathrm{PG}(\bigwedge^n V)$. Let W be the subspace of $\bigwedge^n V$ generated by all 1-spaces $\wedge^n(p)$, where p is a point of Δ. The map $\wedge^n$ then defines a full polarized embedding e_{gr} of Δ into $\mathrm{PG}(W)$ which is called the *Grassmann-embedding of* $DW(2n-1, q)$. The vector-dimension of the Grassmann-embedding is $\binom{2n}{n} - \binom{2n}{n-2}$ (see, e.g., Burau [3, 82.7] and De Bruyn [14]). If $q \neq 2$, the Grassmann-embedding is the absolutely universal embedding of Δ [9,17]. If $q = 2$, the absolutely universal embedding of $DW(2n-1, 2)$ has dimension $\frac{(2^n+1)(2^{n-1}+1)}{3}$ (Li [18], Blokhuis and Brouwer [1]).

Δ also admits a full polarized embedding e_{sp} of vector dimension 2^n, called the *spin-embedding of* $DW(2n-1, q)$. We refer to Buekenhout and Cameron [2] for a description of e_{sp}. By De Bruyn and Pasini [16, Corollary 1.5], it follows that

Theorem 1.1. *Let e be a polarized embedding of a dual polar space of rank n. Then the vector dimension of e is at least 2^n.*

By [5], we have

Theorem 1.2. *Let $\Delta \cong DW(2n-1, q)$, $q = 2^r$. Then the minimal full polarized embedding of Δ is the spin-embedding of Δ (of vector dimension 2^n).*

In this paper we are interested in symplectic dual polar spaces of rank 3 hence from hereon $\Delta := DW(5, 2^r)$. Let $e_{gr} : \Delta \to \Sigma_1$, where $\Sigma_1 = \mathrm{PG}(13, q)$, be the Grassmann-embedding of Δ and let $e_{sp} : \Delta \to \Sigma_2$, where $\Sigma_2 = \mathrm{PG}(7, q)$, be the spin-embedding of Δ. By [5, Theorem 1.4] (see also Cossidente and King [11,12]), the spin-embedding of Δ can be obtained as quotient of the Grassmann-embedding.

In this paper we give a geometrical description of e_{sp} by showing how the natural embedding of $W(5, 2^r)$ is involved in the Grassmann-embedding of $DW(5, 2^r)$. More precisely, we describe the subspace of Σ_1 over which it is possible to take the quotient of e_{gr} and obtain an embedding isomorphic to e_{sp}.

Denote by $\mathcal{M}$ the set of all quads of Δ. Every $M \in \mathcal{M}$ is mapped by the Grassmann-embedding e_{gr} into a non-degenerate quadric $Q_M \cong Q(4, q)$ of rank 2 of a 4-dimensional subspace of Σ_1. Call N_M the nucleus of Q_M. Define the following subspace $\mathcal{N}$ of Σ_1:

$$\mathcal{N} = \left\langle N_M\colon\ N_M \text{ nucleus of } e_{gr}(M) \right\rangle_{M \in \mathcal{M}}. \tag{1}$$

Put $\Pi := W(5, 2^r)$. For any point x of Π, let M_x denote the set of planes of Π through x. Note that M_x, regarded as set of points of Δ, is a quad of Δ. Consider $e_{gr}(M_x)$. Then $e_{gr}(M_x)$ is isomorphic to a non-degenerate parabolic quadric $Q_x := Q_{M_x} \cong Q(4, q)$ naturally embedded in a 4-dimensional subspace of Σ_1. Let N_x be the nucleus of Q_x.

Define a map from the point-set of Π to the subspace $\mathcal{N}$ as follows

$$e_{\mathcal{N}} : \Pi \to \mathcal{N}, \quad e_{\mathcal{N}}(x) = N_x. \tag{2}$$

Clearly, the map $e_{\mathcal{N}}$ is well defined. The following is the main theorem of the paper.

Theorem 1.3.

(1) *Let $q = 2^r$ and $\Delta \cong DW(5, q)$. The spin-embedding of Δ is isomorphic to the quotient of the Grassmann-embedding of $DW(5, q)$ taken over the projective space $\mathcal{N}$.*
(2) *The map $e_{\mathcal{N}}$ realizes the natural embedding of $W(5, 2^r)$ into* $\mathrm{PG}(5, 2^r)$.

2. A full embedding of $W(5, q)$

Henceforth, $\Pi := W(5, 2^r)$, $\Delta := DW(5, 2^r)$, $\Sigma_1 := \mathrm{PG}(13, q)$ and $e_{gr} : \Delta \to \Sigma_1$ is the Grassmann-embedding of Δ. We will often regard Π respectively Δ as point-line geometries $(\mathcal{P}(\Pi), \mathcal{L}(\Pi))$ respectively $(\mathcal{P}(\Delta), \mathcal{L}(\Delta))$.

We will prove that the map $e_{\mathcal{N}}$ defined in (2) is a full embedding of Π (Proposition 2.3). First, let us recall some facts about the structure of the Grassmann-embedding of Δ which will be used in the proof of Proposition 2.3.

Following [4], if p is a point of Δ, we define the *tangent space of index* 1 at the point $e_{gr}(p)$ as follows

$$T_1^{e_{gr}}(p) := \langle e_{gr}\big(\Delta_1^*(p)\big)\rangle. \tag{3}$$

If no confusion is possible, we will write $T_1(p)$ instead of $T_1^{e_{gr}}(p)$ and e instead of e_{gr}. It is possible to define (see [4]) a map e_1^p from the point-set of the residue $Res_\Delta(p) \cong \mathrm{PG}(2, q)$ of p to the point-set of the quotient space $M_1(p) = T_1(p)/p$. (We recall that $Res_\Delta(p)$ is the projective plane formed by the lines and quads of Δ through p.) If F is a line through p, then we define $e_1^p(F) := \langle e(p), e(q)\rangle$, where q is an arbitrary point of F different from p. The following proposition was first stated in Pasini [19] for symplectic dual polar spaces of arbitrary rank n. It was then generalized in Cardinali and De Bruyn [4] to maps e_i^p, $i \in \{1, \dots, n-1\}$. We have adapted it to the case $n = 3$.

Proposition 2.1. *Let e be the Grassmann-embedding of $\Delta = DW(5, q)$ and let p be a point of Δ. Then e_1^p is isomorphic to the veronese map from* $\mathrm{PG}(2, q)$ *to* $\mathrm{PG}(5, q)$.

We are interested in the case q even.

Let ν be the mapping from $\mathrm{PG}(2, q)$ to $\mathrm{PG}(5, q)$ sending the point (X_1, X_2, X_3) to the point $(X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3)$. We call it a (*quadratic*) *veronese map*. The image of ν is a veronese variety $\mathcal{V}$ of $\mathrm{PG}(5, q)$. We will use the following property of the veronese map.

Property 2.2. *Let $q = 2^r$ and ν :* $\mathrm{PG}(2, q) \to \mathrm{PG}(5, q)$ *be a veronese map. Then the image of a pencil of $q + 1$ lines through a point of* $\mathrm{PG}(2, q)$ *is a pencil of conics of* $\mathrm{PG}(5, q)$ *through a point. The $q + 1$ nuclei of these conics form a line of* $\mathrm{PG}(5, q)$.

Proof. Straightforward. □

Let us go back to the map $e_{\mathcal{N}}$ defined in (2).

With abuse of notation we will often identify an element x of Δ with its image $e_{gr}(x)$ in Σ_1. It will be clear from the context what we are referring to. We will also regard $e_{gr}(\Delta)$ as the point-line geometry having as point-set $e_{gr}(\mathcal{P}(\Delta))$ and as line-set $e_{gr}(\mathcal{L}(\Delta))$.

Theorem 2.3. *The map $e_{\mathcal{N}}$ is a full embedding of Π into $\mathcal{N}$.*

Proof. We will first prove that any line l of Π is mapped onto a line of $\mathcal{N}$. Let x be an arbitrary point on l and consider the set of $q+1$ planes of Π through l. The set $\{Q_x\}_{x\in l}$ is a pencil $\mathcal{F}_L$ of $q+1$ parabolic quadrics of rank 2 of Σ_1. All the quadrics Q_x, $x \in l$, pass through the line L arising as image under e_{gr} of the set of planes of Π through l. We shall prove that the set $\{N_x\}_{x\in l} = e_{\mathcal{N}}(l)$ is a line of $\mathcal{N}$.

Take a point z on L and denote by $z^{\perp_x}$ the tangent space to the quadric Q_x at the point z. Then $z^{\perp_x} \cap Q_x$ is a quadratic cone $\mathcal{C}_z^x$ with vertex z and nuclear line n_z^x. It is clear that the nucleus N_x of Q_x belongs to n_z^x. Let z run in L and keep the quadric Q_x fixed. For any $z \in L$, the quadratic cones $\mathcal{C}_z^x$ with vertex z have the property that their nuclear lines n_z^x meet at the common point N_x.

Now fix a point z in L and let the quadric Q_x vary in the pencil $\mathcal{F}_L$. For all $x \in l$, consider the line n_z^x through z. The lines of Δ through z span the tangent space of index 1 at the point z (see definition (3)). By Proposition 2.1, $e^z_{gr_1}$ is isomorphic to the veronese map ν from $\mathrm{PG}(2,q)$ to $\mathrm{PG}(5,q)$ hence $M_1(z) = T_1(z)/z \cong \mathrm{PG}(5,q)$. All the lines of Δ through z (respectively the quadratic cones $\mathcal{C}_z^x$, $x \in l$, with vertex z) can be regarded as points (respectively conics) of the veronese variety $\mathcal{V}$ of $M_1(z)$. In particular, since all cones $\mathcal{C}_z^x$, $x \in l$, share the line L, they can be regarded as a pencil of conics of $\mathcal{V}$ having the point L in common. The nuclei of such conics are precisely the nuclear lines n_z^x of the cones $\mathcal{C}_z^x$. By Property 2.2, the points n_z^x are all collinear. Hence the nuclear lines of the cones $\mathcal{C}_z^x$ span a plane $\pi_z = \langle n_z^x \rangle_{x\in l}$. Clearly, $N_x \in \pi_z$ for every $x \in l$.

If one repeats the same argument for any point $z \in L$, it is immediate to see that $\{N_x\}_{x\in l} \subseteq \pi_z$, $\forall z \in L$. Note that $\pi_z \neq \pi_{z'}$ if $z \neq z'$. Hence $\{N_x\}_{x\in l} = \bigcap\{\pi_z,\ z \in L\}$, i.e. $e_{\mathcal{N}}(l) := \{N_x\}_{x\in l}$ is a line of $\mathcal{N}$.

We will now prove that $e_{\mathcal{N}}$ is injective. Take two arbitrary distinct points x_1 and x_2 of Π. By way of contradiction suppose $e_{\mathcal{N}}(x_1) = e_{\mathcal{N}}(x_2)$, i.e. $N_{x_1} = N_{x_2}$.

First suppose that x_1 and x_2 are collinear in Π. Put $Q_1 := Q_{x_1}$ and $Q_2 := Q_{x_2}$ with nuclei $N_1 := N_{x_1}$ and $N_2 := N_{x_2}$, respectively, and write $\perp_i$ for $\perp_{x_i}$. As x_1 and x_2 are collinear in Π, $Q_1 \cap Q_2$ is a line of Δ. Let $L := Q_1 \cap Q_2$ and take a point $z \in L$. Denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the quadratic cones $z^{\perp_1} \cap Q_1$ and $z^{\perp_2} \cap Q_2$, respectively, and by n_1 and n_2 their nuclear lines, respectively. Since the lines n_1 and n_2 both pass through z and $N_1 = N_2$, then n_1 and n_2 coincide. In $M_1(z) = \mathrm{PG}(5,q)$, the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ appear as two conics through the point L having the same nucleus $n_1 = n_2$. This is impossible because the images of two distinct lines of the plane under the veronese map are two distinct conics through a point with distinct nuclei (this follows from Property 2.2). Hence $e_{\mathcal{N}}(x_1) \neq e_{\mathcal{N}}(x_2)$.

Suppose now that x_1 and x_2 are non-collinear in Π. Let y be a point of Π collinear (in Π) with both x_1 and x_2. Hence $Q_1 \cap Q_y$ is a line L_1 of Δ, $Q_2 \cap Q_y$ is a line L_2 of Δ and $Q_1 \cap Q_2 = \emptyset$, where $Q_1 := Q_{x_1}$ and $Q_2 := Q_{x_2}$. Put $N_1 := N_{x_1}$ and $N_2 := N_{x_2}$ for the nuclei of Q_1 and Q_2 respectively. We are assuming $N_1 = N_2$, hence the line N_1N_y spanned by the nuclei N_1 and N_y, and the line N_2N_y spanned by the nuclei N_2 and N_y, coincide. (Note that $N_1 \neq N_y \neq N_2$ by the previous paragraph.) Denote by n that line. The line n can be seen as $e_{\mathcal{N}}(x_1 y)$, where $x_1 y$ is a singular line of Π, hence each point on n is the nucleus of a quadric through the line L_1. Similarly, every point on n is the nucleus of a quadric through the line L_2. The graph having as vertices the quads and the adjacency relation defined as follows: two vertices Q and Q' are adjacent whenever $Q \cap Q'$ is a line, is connected of diameter 2. By transitivity on ordered pairs of vertices at distance two, we will find at least two intersecting quads with the same nucleus. Impossible by the first part of the lemma. Hence $e_{\mathcal{N}}(x_1) \neq e_{\mathcal{N}}(x_2)$.

We have shown that $e_{\mathcal{N}}$ is an embedding. Since lines of Π and lines of $\mathcal{N}$ have the same number of points, $e_{\mathcal{N}}$ is a full embedding of Π.

The proposition is proved. □

By Tits [23], there exist only two full embeddings of Π. One of them embeds Π as $W(5,q)$ in $\mathrm{PG}(5,q)$ and the other one embeds Π as $Q(6,q)$ in $\mathrm{PG}(6,q)$.

By this remark and Theorem 2.3 we immediately get the following:

Corollary 2.4. *One of the following two cases occurs*:

(1) $e_{\mathcal{N}}: \Pi \to \mathrm{PG}(5,q)$, *i.e.* $\mathcal{N} \cong \mathrm{PG}(5,q)$ *and* $e_{\mathcal{N}}(\Pi) = W(5,q)$;
(2) $e_{\mathcal{N}}: \Pi \to \mathrm{PG}(6,q)$, *i.e.* $\mathcal{N} \cong \mathrm{PG}(6,q)$ *and* $e_{\mathcal{N}}(\Pi) = Q(6,q)$.

3. Proof of Theorem 1.3

Let $\mathcal{N}$ be the subspace of Σ_1 defined in (1) of Section 1.2 and $e_{gr}: \Delta \to \Sigma_1$ be the Grassmann-embedding of Δ. Since e_{gr} is a homogeneous embedding, every automorphism of Δ lifts to an automorphism of Σ_1. Denote by G the group induced in Σ_1 by $Aut(\Delta) \cong Aut(Sp(6,q)) \cong P\Gamma O(7,q)$. Clearly, G stabilizes $\mathcal{N}$ and $e_{\mathcal{N}}(\Pi)$, acting as $Aut(Sp(6,q))$ or $P\Gamma O(7,q)$ on $e_{\mathcal{N}}(\Pi)$, according to whether case (1) or (2) of Corollary 2.4 occurs. Recall the following facts about elements of Δ: (1) $\langle e_{gr}(Q)\rangle \cap e_{gr}(\Delta) = e_{gr}(Q)$ for any quad Q of Δ; (2) if Q_1 and Q_2 are two disjoint quads of Δ, the projection of Q_1 onto Q_2 is an isomorphism between Q_1 and Q_2.

Recall that we often identify an element of Δ with its image in Σ_1. According to this, we will identify a quad of Δ with its image, thus regarding it as a substructure of $e_{gr}(\Delta)$. It is also clear that $\mathrm{d}_\Delta(x_1, x_2) = k$ if and only if the induced distance $\mathrm{d}(x_1, x_2)$ in $e_{gr}(\Delta)$ between x_1 and x_2 is k.

Next lemma shows that it is possible to take a quotient of e_{gr} over $\mathcal{N}$.

Lemma 3.1. *Both the following hold*:

(1) $\langle \mathcal{N}, e_{gr}(p)\rangle \neq \mathcal{N}$ *for every point* p *of* Δ;
(2) $\langle \mathcal{N}, e_{gr}(p_1)\rangle \neq \langle \mathcal{N}, e_{gr}(p_2)\rangle$ *for any two distinct points* p_1 *and* p_2 *of* Δ.

Proof. (1) Suppose by way of contradiction that there exists a point $p \in \Delta$ such that $\langle \mathcal{N}, e_{gr}(p)\rangle = \mathcal{N}$. The point $e_{gr}(p)$ cannot be the nucleus of any quadric (by fact (1) recalled at the beginning of this section), i.e. cannot be a singular point of $\mathcal{N}$. Hence, by Corollary 2.4, $\mathcal{N} \cong \mathrm{PG}(6,q)$ and $e_{\mathcal{N}}$ realizes Π as $Q(6,q)$ in $\mathcal{N}$. Since the group G is transitive on the points of $e_{gr}(\Delta)$, we have $\{p^\sigma \colon \sigma \in G\} = e_{gr}(\Delta)$. On the other hand, $e_{gr}(p) \in \mathcal{N}$ by assumption, and

G stabilizes $\mathcal{N}$. Hence $e_{gr}(\Delta) \subseteq \mathcal{N}$. Clearly impossible. So $\mathcal{N}$ is disjoint from $e_{gr}(\Delta)$ and (1) is proved.

(2) We will show that if p_1, p_2 are two distinct points in $e_{gr}(\Delta)$, then the projective line $l = p_1p_2$ spanned by p_1 and p_2 does not intersect $\mathcal{N}$. By way of contradiction, suppose that $l \cap \mathcal{N} = P$, a point of $\mathcal{N}$.

$\boxed{\textit{Case}\ \text{(a)}}$ The point P is a singular point of $\mathcal{N}$, i.e. there exists a quad of $e_{gr}(\Delta)$ having P as nucleus. There are three cases to consider according to the distance between p_1 and p_2:

Case a1: $d_\Delta(p_1, p_2) = 1$. Then the line $l = p_1p_2$ is contained in $e_{gr}(\Delta)$, hence $P \in \mathcal{N} \cap e_{gr}(\Delta)$. Impossible by (1).

Case a2: $d_\Delta(p_1, p_2) = 2$. Let Q be the (unique) quad of Δ containing p_1 and p_2. The 4-dimensional projective space $\langle Q\rangle$ spanned by Q intersects $\mathcal{N}$ in the nucleus N_Q of Q because $N_Q \in \mathcal{N}$ by definition of $\mathcal{N}$ (and clearly $N_Q \in \langle Q\rangle$). If X was a point different from N_Q in $\langle Q\rangle \cap \mathcal{N}$, then the line $N_QX \subseteq \mathcal{N}$ would contain a point of $e_{gr}(\Delta)$. Impossible by (1). Hence $\langle Q\rangle \cap \mathcal{N} = N_Q$, therefore $N_Q = P = l \cap \mathcal{N}$. This is not possible because the line l is secant to Q and passes through the nucleus N_Q of Q.

Case a3: $d_\Delta(p_1, p_2) = 3$. Take the stabilizer G_{p_1,p_2} of p_1 and p_2 in G. Then G_{p_1,p_2} fixes P. Since P is the nucleus of a certain quad Q, G_{p_1,p_2} stabilizes Q. This is impossible because the stabilizer in G of two points of Δ at distance 3 does not fix any quad.

$\boxed{\textit{Case}\ \text{(b)}}$ The point $P = l \cap \mathcal{N}$ is a non-singular point of $\mathcal{N}$.

By Corollary 2.4, $\mathcal{N} \cong \mathrm{PG}(6, q)$ and Π is realized as $Q(6, q)$ in $\mathcal{N}$. Let N_0 be the nucleus of $e_{\mathcal{N}}(\Pi) \cong Q(6, q)$.

Case b1: $P \neq N_0$. The line through P and N_0 is a tangent line to $Q(6, q)$ at a point N_Q which is the nucleus of a certain quad Q of Δ. The stabilizer G_{p_1,p_2} of p_1 and p_2 in G fixes the points P and the quadric $e_{\mathcal{N}}(\Pi) \subseteq \mathcal{N}$. Hence G_{p_1,p_2} fixes N_0 and P, so it fixes N_Q. Then the quad Q is also stabilized by G_{p_1,p_2}. This is not possible if $d_\Delta(p_1, p_2) = 1, 3$. The remaining case to rule out is $d_\Delta(p_1, p_2) = 2$ with Q the quad containing the points p_1 and p_2. The 4-dimensional projective space $\langle Q\rangle$ intersects $\mathcal{N}$ only in the nucleus N_Q of Q. Therefore the line p_1p_2 intersects $\mathcal{N}$ in N_Q. Hence $P = N_Q$ is singular, contradiction.

Case b2: $P = N_0$. The points p_1 and p_2 can not be at distance 1 since $\mathcal{N} \cap e_{gr}(\Delta) = \emptyset$ by part (1) of the lemma. Hence, $d_\Delta(p_1, p_2) = 2$ or 3. Take a point $p_3 \in e_{gr}(\Delta)$ such that $d_\Delta(p_1, p_2) = d_\Delta(p_2, p_3)$, $p_3 \neq p_1$. The group G is transitive on the ordered pairs of points at distance 2 and 3, respectively. So, $\exists\sigma \in G$ such that $(p_1, p_2)^\sigma = (p_3, p_2)$ and $N_0^\sigma = N_0$. The (distinct) lines p_1p_2 and p_2p_3 both pass through p_2 and N_0. Impossible.

The lemma is proved. □

Lemma 3.2. *The quotient $e_{gr}/\mathcal{N}$ is a full polarized embedding of Δ.*

Proof. By Lemma 3.1, the quotient embedding $e_{gr}/\mathcal{N}$ exists and is well defined. By [5, Lemma 1.1], $e_{gr}/\mathcal{N}$ is a full embedding of Δ.

By way of contradiction suppose that $e_{gr}/\mathcal{N}$ is not a polarized embedding, i.e. $\langle (e_{gr}/\mathcal{N})(H_x)\rangle = \Sigma/\mathcal{N}$, for a singular hyperplane H_x of Δ. Hence $\langle \mathcal{N}, \langle e_{gr}(H_x)\rangle\rangle = \Sigma$ which implies that $\mathcal{N} \cap \langle e_{gr}(H_x)\rangle$ is a hyperplane of $\mathcal{N}$. Indeed, $\langle e_{gr}(H_x)\rangle$ is a hyperplane of Σ since e_{gr} is a polarized embedding. We claim the following

(∗) If p_1 and p_2 are two distinct points of Δ then $H_1 := \langle e_{gr}(H_{p_1})\rangle \cap \mathcal{N} \neq \langle e_{gr}(H_{p_2})\rangle \cap \mathcal{N} =: H_2$.

Proof of claim $(*)$. Put $\delta = \mathrm{d}_\Delta(p_1, p_2)$, $\delta = 1, 2$ or 3. By way of contradiction suppose $H_1 = H_2$. The group G is transitive on the pairs of points of Δ at distance δ. Moreover, the graph defined on the point-set of Δ by the relation '*being at distance* δ' is connected. It follows that we have $e_{gr}(H_x) \cap \mathcal{N} = e_{gr}(H_y) \cap \mathcal{N}$ for any two points x, y of Δ, no matter what $\mathrm{d}_\Delta(x, y)$ is. So, $\langle e_{gr}(H_x)\rangle \cap \mathcal{N} = H$ for a given hyperplane H of $\mathcal{N}$ and any given point x of Δ. Since the nucleus $N_Q \in \langle e_{gr}(H_x)\rangle \cap \mathcal{N}$ for every point x of Δ and every quad through x, $N_Q \in H$ for every quad Q of Δ. Regarding Q as a point p of Π, this would force $\langle e_{\mathcal{N}}(p)\rangle_{p\in\Pi}$ to span a hyperplane section of $\mathcal{N}$. Impossible.

Claim $(*)$ follows.

If $e_{gr}/\mathcal{N}$ was not a polarized embedding of Δ, then by $(*)$, the number of singular hyperplanes of Δ would be less than or equal to the number of hyperplanes of $\mathcal{N}$. But the number of singular hyperplanes of Δ is $q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1$ and the number of hyperplanes of $\mathcal{N}$ is $q^5 + q^4 + q^3 + q^2 + q + 1$ or $q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$ according whether Case (1) or Case (2) of Corollary 2.4 occurs. Hence, $e_{gr}/\mathcal{N}$ is a polarized embedding of Δ. □

Theorem 3.3. $e_{gr}/\mathcal{N} \cong e_{sp}$.

Proof. By Lemma 3.2, $e_{gr}/\mathcal{N}$ is a full polarized embedding of Δ. By Corollary 2.4, either $\mathcal{N} = \mathrm{PG}(5, q)$ or $\mathcal{N} = \mathrm{PG}(6, q)$. Hence the projective dimension of $e_{gr}/\mathcal{N}$ is 6 or 7. By Theorem 1.1, the projective dimension of $e_{gr}/\mathcal{N}$ cannot be 6. Hence $e_{gr}/\mathcal{N}$ is a minimal polarized full embedding of Δ. By Theorem 1.2, $e_{gr}/\mathcal{N}$ is isomorphic to the spin-embedding of Δ, that is $e_{gr}/\mathcal{N} \cong e_{sp}$. □

Theorem 3.3 is exactly part (1) of Theorem 1.3. By Theorem 3.3 we have $\mathcal{N} = \mathrm{PG}(5, q)$ hence, by Corollary 2.4, part (2) of Theorem 1.3 follows.

4. Remarks

Throughout the paper we have assumed to deal with finite fields of characteristic 2. Perhaps the same result holds also in the case of infinite perfect fields of characteristic 2. Note that finiteness has been exploited in an essential way only at the end of the proof of Lemma 3.2.

A generalization to the case of rank $n > 3$ is likely to be possible, at least in principle. The main difficulty to overcome is that, when Π has rank $n > 3$ we need complete control over all full embeddings of its Grassmannians of $(n-3)$-dimensional singular subspaces. This information does not seem to be available yet.

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